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ON T-DUALITY AND SUPERSYMMETRY[†]

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ABSTRACT

The interplay between T-duality and supersymmetry in string theory is explored. It is shown that T-duality is always compatible with supersymmetry and simply changes a local realization to a non-local one and vice versa. Non-local realizations become natural using classical parafermions of the underlying conformal field theory. Examples presented include hyper-kähler metrics and the backgrounds for the $SU(2) \otimes U(1)$ and $SU(2)/U(1) \otimes U(1) \otimes U(1)$ exact conformal field theories.

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1. Introduction

This note is based on a lecture given in the Leuven conference on Gauge Theories, Applied Supersymmetry and Quantum Gravity and contains some of our recent work on Target space duality (T-duality) and supersymmetry [1,2]. Further details can be found in there and other papers that are directly related to the subject [3-6]. Special care was taken so that the note is as introductory and self contained as possible.

The description of certain phenomena in terms of effective field theories might lead to paradoxes. One such case is the apparent incompatibility between T-duality and supersymmetry which was first noticed in [3] and subsequently with other examples in [4]. These obstructions were observed completely at the classical level for the case of extended world-sheet supersymmetry and at 1-loop level in α' for target space supersymmetry. T-duality is a stringy property and provides an equivalence between strings propagating in different backgrounds [7] and as such it should not lead to a real breaking of other genuine symmetries such as supersymmetry. The natural explanation of this paradox is that, non-local world-sheet effects associated with the T-duality transformation replace a local realization of supersymmetry with a non-local one, but never break supersymmetry which is always manifest at the conformal field theory (CFT) level [1]. This point of view was also advocated in [5,6].

This issue is relevant for string phenomenology since if duality breaks or, restores manifest supersymmetry [6,2] (we will use the terms manifest and locally realized supersymmetry in an equivalent way), then this phenomenon should be incorporated in supersymmetry breaking scenarios. “Apparently” non-supersymmetric backgrounds can qualify as vacuum solutions to superstring theory when it is possible to restore manifest supersymmetry through non-local world-sheet effects at the string level. In addition, it raises the possibility that various solutions that are of interest in black hole physics or cosmology might have hidden supersymmetries in a string context. Since this necessarily involves non-local world-sheet effects it is important in our effort to understand the way string theory could resolve fundamental problems in physics.

The rest of this section is devoted to the formulation of abelian T-duality as a canonical transformation. In section 2 extended world-sheet supersymmetry and its relation to T-duality is explored. As examples we consider hyper-kahler manifolds and the semi-wormhole exact solution of 4-dim string theory. In the later case direct contact with the underlying CFT theory and classical parafermions is made. We end this note with section 3 where some related topics are briefly discussed.

T – Duality and canonical transformations: The mechanism by which abelian T-duality changes local realizations of supersymmetry to non-local ones is most transparent in its formulation as a canonical transformation [8]. Here we follow a slightly different route which especially suits our purposes. The classical propagation of strings in a general target space with metric $G_{\mu\nu}(X)$ and antisymmetric tensor field $B_{\mu\nu}(X)$ is described by the 2–

dim σ -model Lagrangian density $\mathcal{L} = Q_{\mu\nu}^+ \partial_+ X^\mu \partial_- X^\nu$, where $Q_{\mu\nu}^\pm \equiv G_{\mu\nu} \pm B_{\mu\nu}$. The natural time coordinate on the world-sheet is $\tau = \sigma^+ + \sigma^-$. Also will use $\sigma = \sigma^+ - \sigma^-$ to denote the position along the string for fixed τ . We assume there exist a Killing symmetry associated with the vector field $\partial/\partial X^0$ and we denote the rest of the coordinates by X^i , $i = 1, \dots, d-1$. For convenience we will work in the adapted coordinate system where the background fields are independent of X^0 . The canonical approach [8] starts by computing the conjugate momenta to X^μ , $P_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}$ (the dot denotes derivative with respect to τ)

$$P_\mu = \frac{1}{2} Q_{\mu\nu}^+ \partial_- X^\nu + \frac{1}{2} Q_{\mu\nu}^- \partial_+ X^\nu . \quad (1.1)$$

The transformations $P_0 = \partial_\sigma \tilde{X}^0$, $\partial_\sigma X^0 = \tilde{P}_0$ and the redefinitions $X^i = \tilde{X}^i$, $P_i = \tilde{P}_i$ preserve the Poisson bracket $\{X^\mu(\tau, \sigma), P_\nu(\tau, \sigma')\} = \delta^\mu_\nu \delta(\sigma - \sigma')$ and therefore constitute a canonical transformation. The world-sheet derivatives

$$\partial_\pm X^0 \equiv \dot{X}^0 \pm \partial_\sigma X^0 = G_{00}^{-1} (P_0 - \frac{1}{2} Q_{i0}^+ \partial_+ X^i - \frac{1}{2} Q_{i0}^- \partial_- X^i) \pm \partial_\sigma X^0 , \quad (1.2)$$

transform as well. After some algebra and using the analogous to (1.1) expression for the conjugate momentum to \tilde{X}^0 , \tilde{P}_0 , we obtain

$$\begin{aligned} \partial_\pm X^0 = & \pm \frac{1}{2} (\tilde{G}_{00} + G_{00}^{-1}) \partial_\pm \tilde{X}^0 \pm \frac{1}{2} (\tilde{G}_{00} - G_{00}^{-1}) \partial_\mp \tilde{X}^0 \\ & \pm \frac{1}{2} (\tilde{Q}_{i0}^\pm \mp G_{00}^{-1} Q_{i0}^\pm) \partial_\pm X^i \pm \frac{1}{2} (\tilde{Q}_{0i}^\pm \mp G_{00}^{-1} Q_{0i}^\pm) \partial_\mp X^i . \end{aligned} \quad (1.3)$$

In order to preserve 2-dimensional Lorentz invariance we should have world-sheet derivatives of the same chirality in both sides of (1.3). Setting to zero the relevant terms we relate the background fields in the transformed model to those in the original one

$$\tilde{G}_{00} = G_{00}^{-1} , \quad \tilde{Q}_{0i}^\pm = \pm G_{00}^{-1} Q_{0i}^\pm , \quad \tilde{Q}_{ij}^\pm = Q_{ij}^\pm - G_{00}^{-1} Q_{i0}^+ Q_{0j}^+ . \quad (1.4)$$

These are nothing but Buscher's duality transformation rules [7]. Actually, the transformation of Q_{ij}^+ follows by identifying the coefficients of world-sheet derivatives in $P_i = \tilde{P}_i$ after 2-dimensional Lorentz invariance was taken into account. The conformal invariance also requires that the corresponding dilaton field Φ is shifted by $\ln G_{00}$ [7]. The transformed world-sheet derivatives under duality then become

$$\partial_\pm X^0 = \pm G_{00}^{-1} (\partial_\pm \tilde{X}^0 \mp Q_{i0}^\pm \partial_\pm X^i) = \pm \tilde{Q}_{\mu 0}^\pm \partial_\pm \tilde{X}^\mu . \quad (1.5)$$

This transformation amounts to a non-local redefinition of the target space variable associated with the Killing symmetry,

$$X^0 = \int \tilde{Q}_{\mu 0}^+ \partial_+ \tilde{X}^\mu d\sigma^+ - \tilde{Q}_{\mu 0}^- \partial_- \tilde{X}^\mu d\sigma^- . \quad (1.6)$$

Despite the non-localities, the dual target space fields (1.4) are locally related to the original ones. However, other geometrical objects in the target space, are not bound to be always local in the dual picture [1]. This will be discussed extensively in the next section.

$N = 1$ world – sheet supersymmetry under duality: It is well known that any background can be made $N = 1$ supersymmetric [9]. Thus we do not expect any clash with duality (abelian and non-abelian T-duality as well as S-duality) in this case. In fact one can formulate abelian duality in a manifestly supersymmetric way by using $N = 1$ superfields [5]. The result is that the transformation of $\partial_{\pm} X^0$ is given by (1.5) plus a quadratic term in the fermions and that the fermions themselves transform as the world-sheet derivatives in (1.5). In other words bosons in the dual model are composites of bosons and fermions of the original model. This boson-fermion symphysis was first observed in [10] for the supersymmetric extension of the Chiral Model on $O(4)$ and its non-abelian dual. For convenience we will not subsequently write the dependence on the world-sheet fermions since, on general grounds, it is completely dictated by the purely bosonic term [2].

2. T-Duality and extended world-sheet supersymmetry

In contrast with $N = 1$, it is well known that extended $N = 2$ supersymmetry [11,12,13] requires that the background is such that an (almost) complex (hermitian) structure $F_{\mu\nu}^{\pm}$ in each sector, associated to the right and left-handed fermions, exists. Similarly, $N = 4$ extended supersymmetry [12,13,14] requires that, in each sector, there exist three complex structures $(F_I^{\pm})_{\mu\nu}$, $I = 1, 2, 3$. The complex structures are covariantly constant, with respect to generalized connections that include the torsion, antisymmetric matrices and in the case of $N = 4$ they obey the $SU(2)$ Clifford algebra. These conditions put severe restrictions on the backgrounds that admit a solution. For instance in the absence of torsion the metric should be Kahler for $N = 2$ and hyper-kahler for $N = 4$ [12].

We are interested in finding the duality transformation properties of the complex structures. We simply examine the 2-forms defined in each chiral sector separately [1]

$$F_I^{\pm} = (F_I^{\pm})_{\mu\nu} dX^{\mu} \wedge dX^{\nu} = 2(F_I^{\pm})_{0i} dX^0 \wedge dX^i + (F_I^{\pm})_{ij} dX^i \wedge dX^j . \quad (2.1)$$

In order to find the correct transformation properties under T-duality, we simply have to use the replacement $dX^{\mu} \rightarrow \partial_+ X^{\mu}$ for F_I^+ and $dX^{\mu} \rightarrow \partial_- X^{\mu}$ for F_I^- . This is only meant to be a prescription for extracting the relevant part of the complex structures under the duality transformation (1.5). Then the dual complex structures in component form are

$$(\tilde{F}_I^{\pm})_{0i} = \pm G_{00}^{-1} (F_I^{\pm})_{0i} , \quad (\tilde{F}_I^{\pm})_{ij} = (F_I^{\pm})_{ij} + G_{00}^{-1} ((F_I^{\pm})_{0i} Q_{j0}^{\pm} - (F_I^{\pm})_{0j} Q_{i0}^{\pm}) . \quad (2.2)$$

There was a crucial assumption made implicitly, namely that in the adapted coordinate system the complex structures F_I^{\pm} were independent of the Killing coordinate X^0 similarly

to background fields. It turns out that this is a correct assumption for the case of $N = 2$ extended world-sheet supersymmetry. However, it is not always true in the case of $N = 4$ extended supersymmetry where in certain cases two of the complex structures depend on X^0 explicitly and they form an $SO(2)$ doublet. Then, under duality they become non-local due to the mechanism associated with (1.6). A useful criterion for when abelian T-duality preserves manifest $N = 4$ extended world-sheet (and target space) supersymmetry is [2]¹

$$\partial_\mu Q_{0\nu}^\mp (F_I^\pm)^{\mu\nu} = 0, \quad I = 1, 2, 3. \quad (2.3)$$

In the example that we consider next, in which local $N = 4$ is preserved under duality, one can explicitly show that (2.3) is indeed true for all three complex structures. In contrast, in the rest of the examples it is violated for $I = 3$, in agreement with the fact that T-duality in these cases destroys manifest $N = 4$, which then is realized non-locally.

2.1. Hyper-Kähler manifolds and T-duality

Let us first consider 4-dim pure gravitational backgrounds with $N = 4$ extended supersymmetry, which are known to be hyper-kähler self-dual manifolds, that in addition have one Killing symmetry. A complete classification of them exists and depends on whether or not the covariant derivative of the corresponding Killing vector is self-dual [15]. Accordingly, the Killing vector is of the translational or the rotational type.

The translational case:

In the case of 4-dim hyper-Kähler manifolds with a translational symmetry the metric assumes the form [16]

$$ds^2 = V(dT + \Omega_i dx^i)^2 + V^{-1} dX_i dX^i, \quad (2.4)$$

where $T \equiv X^0$ is the coordinate adapted for the translational Killing vector field $\partial/\partial T$. Moreover, Ω_i are constrained to satisfy the special conditions

$$\partial_i V^{-1} = \epsilon_{ijk} \partial_j \Omega_k, \quad (2.5)$$

as a result of the self-dual character of the metric. It also follows that V^{-1} satisfies the 3-dim flat space Laplace equation. Localized solutions of this equation correspond to the familiar series of multi-centre Eguchi-Hanson gravitational instantons or to the multi-Taub-NUT family, depending on the asymptotic conditions on V^{-1} (see, for instance, [17] and references therein). The three independent complex structures are [18]

$$F^i = (dT + \Omega_j dX^j) \wedge dX^i - \frac{1}{2} V^{-1} \epsilon^{ijk} dX^j \wedge dX^k. \quad (2.6)$$

¹ It has been used to prove that a marginal deformation by current-bilinears in the Cartan subalgebra of a WZW model for a general quaternionic group always breaks their manifest $N = 4$ world-sheet supersymmetry [2].

We see that the complex structures are invariant under constant shifts of T . Thus under duality $N = 4$ supersymmetry will remain locally realized. Indeed the dual to (2.4) background is found using (1.4) to be

$$\begin{aligned} d\tilde{s}^2 &= V^{-1}(d\tilde{T}^2 + dX_i dX^i) , \\ \tilde{B} &= 2 \Omega_i d\tilde{T} \wedge dX^i , \quad \tilde{\Phi} = \ln V , \end{aligned} \quad (2.7)$$

where we have denoted by \tilde{T} the Killing coordinate in the dual model. For the dual complex structures we use (2.2) and find the result [1]

$$\tilde{F}^i = V^{-1}(\pm d\tilde{T} \wedge dX^i - \frac{1}{2}\epsilon^{ijk} dX^j \wedge dX^k) , \quad (2.8)$$

which define a local realization of the $N = 4$ world-sheet supersymmetry for (2.7).

The rotational case:

In the case of 4-dim hyper-Kähler manifolds with a rotational Killing symmetry, there exists a coordinate system (τ, x, y, z) in which the corresponding line element assumes the form

$$ds^2 = v(d\tau + \omega_1 dx + \omega_2 dy)^2 + v^{-1}(e^\Psi dx^2 + e^\Psi dy^2 + dz^2) . \quad (2.9)$$

In these adapted coordinates the rotational Killing vector field is $\partial/\partial\tau$ and all the components of the metric are expressed in terms of a single scalar field $\Psi(x, y, z)$ [15], so that $v^{-1} = \partial_z \Psi$, $\omega_1 = -\partial_y \Psi$, $\omega_2 = +\partial_x \Psi$ and where $\Psi(x, y, z)$ satisfies the continual Toda equation

$$(\partial_x^2 + \partial_y^2)\Psi + \partial_z^2 e^\Psi = 0 . \quad (2.10)$$

Examples of metrics which can be put into the form (2.9) include the Eguchi–Hanson instanton and the Taub–NUT and Atiyah–Hitchin metrics on the moduli space of the $SU(2)$ 2-monopole solutions in the BPS limit. In fact, the first two metrics exhibit both kinds of isometries, translational and rotational (see for instance [18]).

Metrics with rotational Killing symmetry differ from those with translational symmetry in that not all three independent complex structures can be chosen to be τ -shift invariant. In fact, only one complex structure can be chosen to be an $SO(2)$ singlet, while the other two necessarily form an $SO(2)$ doublet. We have explicitly [1]

$$F_3 = (d\tau + \omega_1 dx + \omega_2 dy) \wedge dz + v^{-1} e^\Psi dx \wedge dy , \quad (2.11)$$

for the singlet and

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = e^{\frac{1}{2}\Psi} \begin{pmatrix} \cos \frac{\tau}{2} & \sin \frac{\tau}{2} \\ \sin \frac{\tau}{2} & -\cos \frac{\tau}{2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} , \quad (2.12)$$

for the doublet, where

$$f_1 = (d\tau + \omega_2 dy) \wedge dx - v^{-1} dz \wedge dy , \quad f_2 = (d\tau + \omega_1 dx) \wedge dy + v^{-1} dz \wedge dx . \quad (2.13)$$

The T-duality transformation with respect to $X^0 \equiv \tau$, yields the background

$$\begin{aligned} d\tilde{s}^2 &= v^{-1}(e^\Psi dx^2 + e^\Psi dy^2 + dz^2 + d\tilde{\tau}^2) , \\ \tilde{B} &= 2 d\tilde{\tau} \wedge (\omega_1 dx + \omega_2 dy) , \quad \tilde{\Phi} = \ln v . \end{aligned} \quad (2.14)$$

The complex structure (2.11) will remain local in the dual picture, assuming the form

$$\tilde{F}_3^\pm = v^{-1}(\pm d\tilde{\tau} \wedge dz + e^\Psi dx \wedge dy) . \quad (2.15)$$

In contrast, the complex structures forming the doublet (2.12) become non-local and are given by

$$\begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix} = e^{\frac{1}{2}\Psi} \begin{pmatrix} \cos \frac{\tau}{2} & \sin \frac{\tau}{2} \\ \sin \frac{\tau}{2} & -\cos \frac{\tau}{2} \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} , \quad (2.16)$$

where the 2-forms dual to (2.13) are

$$\tilde{f}_1^\pm = v^{-1}(\pm d\tilde{\tau} \wedge dx - dz \wedge dy) , \quad \tilde{f}_2^\pm = v^{-1}(\pm d\tilde{\tau} \wedge dy + dz \wedge dx) . \quad (2.17)$$

The non-localities are due to the explicit dependence on τ which is not a variable in the dual model but rather a functional of the dual variables

$$\tau = \int (v^{-1} \partial_+ \tilde{\tau} - \omega_1 \partial_+ x - \omega_2 \partial_+ y) d\sigma^+ - (v^{-1} \partial_- \tilde{\tau} + \omega_1 \partial_- x + \omega_2 \partial_- y) d\sigma^- . \quad (2.18)$$

Moreover, \tilde{F}_1^\pm and \tilde{F}_2^\pm are not covariantly constant on-shell [1,5]. Nevertheless, they can still be used to define a supersymmetry. We found that the local realization of $N = 4$ world-sheet supersymmetry breaks down to $N = 2$, with \tilde{F}_3^\pm providing the relevant pair of complex structures. At the level of the corresponding superconformal field theory, part of the $N = 4$ world-sheet supersymmetry will be realized with operators that have parafermionic (non-local) behavior. We do not have an exact conformal field theory description of the string gravitational background (2.9), in order to illustrate this point in all generality. For this reason we will examine the question in the special case of a 4-dim semi-wormhole solution and its rotational dual background, where an exact description is available in terms of the $SU(2) \otimes U(1)$ WZW model and its derivatives.

2.2. Complex structures and parafermions

A semi-wormhole solution of 4-dim string theory provides an exact conformal field theory background with $N = 4$ world-sheet supersymmetry [19,20]. The $N = 4$ superconformal algebra can be locally realized in terms of four bosonic currents, three non-Abelian $SU(2)_k$ currents and one Abelian current with background charge $Q = \sqrt{2/(k+2)}$, so that the central charge is $\hat{c} = 4$. There are also four free-fermion superpartners and the solution

is described by the $SU(2)_k \otimes U(1)_Q$ supersymmetric WZW model.² The background fields of this model are given by

$$\begin{aligned} ds^2 &= d\rho^2 + d\varphi^2 + \sin^2 \varphi d\psi^2 + \cos^2 \varphi d\tau^2 , \\ B_{\tau\psi} &= \cos^2 \varphi , \quad \Phi = 2\rho . \end{aligned} \quad (2.19)$$

The analysis that follows was essentially performed in [1] in a slightly different parametrization. Among the three complex structures one is an $SO(2)$ singlet

$$F_3^\pm = d\rho \wedge (\cos^2 \varphi d\tau \pm \sin^2 \varphi d\psi) + \frac{1}{2} \sin 2\varphi (d\tau \mp d\psi) \wedge d\varphi , \quad (2.20)$$

and the other two form an $SO(2)$ doublet

$$\begin{pmatrix} F_1^\pm \\ F_2^\pm \end{pmatrix} = \begin{pmatrix} \cos(\tau \pm \psi) & \sin(\tau \pm \psi) \\ -\sin(\tau \pm \psi) & \cos(\tau \pm \psi) \end{pmatrix} \begin{pmatrix} f_1^\pm \\ f_2^\pm \end{pmatrix} , \quad (2.21)$$

with the definitions

$$\begin{aligned} f_1^\pm &= -d\rho \wedge d\varphi \pm \frac{1}{2} \sin 2\varphi d\tau \wedge d\psi \\ f_2^\pm &= -\frac{1}{2} \sin 2\varphi d\rho \wedge (d\tau \mp d\psi) + (\cos^2 \varphi d\tau \pm \sin^2 \varphi d\psi) \wedge d\varphi . \end{aligned} \quad (2.22)$$

T-duality corresponding to the Killing vector $\partial/\partial\psi$ gives the background [22]

$$\begin{aligned} d\tilde{s}^2 &= d\varphi^2 + \cot^2 \varphi d\alpha^2 + d\beta^2 + d\rho^2 , \\ \tilde{\Phi} &= 2\rho + \ln(\sin^2 \varphi) , \end{aligned} \quad (2.23)$$

with zero antisymmetric tensor, which corresponds to the $SU(2)_k/U(1) \otimes U(1) \otimes U(1)_Q$ model, and where the redefinitions $\alpha = \tilde{\psi} - \frac{\tau}{2}$ and $\beta = \tilde{\psi} + \frac{\tau}{2}$ have been made. Under the T-duality the dual complex structure to F_3^\pm becomes

$$\tilde{F}_3 = d\rho \wedge d\beta + \cot \varphi d\varphi \wedge d\alpha , \quad (2.24)$$

and defines a local supersymmetry. Since there is no torsion there is no distinction between the + and the - components. Similarly the 2-forms (2.22) become

$$\tilde{f}_1 = -d\rho \wedge d\varphi - \cot \varphi d\alpha \wedge d\beta , \quad \tilde{f}_2 = \cot \varphi d\rho \wedge d\alpha + d\beta \wedge d\varphi . \quad (2.25)$$

However, the dual complex structures become non-local due to the explicit appearance of ψ in (2.21). Using the non-local relation to the dual model variables

$$\psi = \int (\cot^2 \varphi \partial_+ \alpha + \partial_+ \beta) d\sigma^+ - (\cot^2 \varphi \partial_- \alpha + \partial_- \beta) d\sigma^- , \quad (2.26)$$

² The realization of the $N = 4$ superconformal algebra in terms of $SU(2)$ currents was first considered in [21].

we write the dual complex structures in the suggestive forms [1]³

$$\begin{aligned}\tilde{F}_1^+ &= \Psi_+ \wedge (d\rho + i\beta) + \Psi_- \wedge (d\rho - id\beta), \quad \tilde{F}_2^+ = i\Psi_+ \wedge (d\rho + i\beta) - i\Psi_- \wedge (d\rho - id\beta), \\ \tilde{F}_1^- &= \bar{\Psi}_+ \wedge (d\rho - i\beta) + \bar{\Psi}_- \wedge (d\rho + id\beta), \quad \tilde{F}_2^- = -i\bar{\Psi}_+ \wedge (d\rho - i\beta) + i\bar{\Psi}_- \wedge (d\rho + id\beta),\end{aligned}\tag{2.27}$$

where the parafermionic-type 1-forms are defined as

$$\Psi_{\pm} = (d\varphi \pm i \cot \varphi d\alpha) e^{\pm i(\beta - \alpha + \psi)}, \quad \bar{\Psi}_{\pm} = (d\varphi \mp i \cot \varphi d\alpha) e^{\pm i(\alpha - \beta + \psi)}, \tag{2.28}$$

and are non-local due to (2.26). They have a natural decomposition in terms of $(1, 0)$ and $(0, 1)$ forms on the string world-sheet

$$\Psi_{\pm} = \Psi_{\pm}^{(1,0)} d\sigma^+ + \Psi_{\pm}^{(0,1)} d\sigma^-, \quad \bar{\Psi}_{\pm} = \bar{\Psi}_{\pm}^{(1,0)} d\sigma^+ + \bar{\Psi}_{\pm}^{(0,1)} d\sigma^-. \tag{2.29}$$

It can be easily verified using the classical equations of motion for the model (2.23) that the chiral and anti-chiral conservation laws

$$\partial_- \Psi_{\pm}^{(1,0)} = 0, \quad \partial_+ \bar{\Psi}_{\pm}^{(0,1)} = 0, \tag{2.30}$$

are obeyed. In fact in this case $\Psi_{\pm}^{(1,0)}$ and $\bar{\Psi}_{\pm}^{(0,1)}$ are nothing but the classical parafermions for the $SU(2)_k/U(1)$ coset with the field β actually providing the dressing to the full 4-dim model $SU(2)_k/U(1) \otimes U(1) \otimes U(1)_Q$. Thus the original local $N = 4$ world-sheet supersymmetry breaks to a local part corresponding to (2.24), and the rest is realized non-locally using the non-local complex structures (2.27). At the (super)CFT level this is manifested by a replacement of the three $SU(2)_k$ currents by two $SU(2)_k/U(1)$ parafermions and one free boson in the realization of the $N = 4$ superconformal algebra [20].

3. Further developments and comments

In this section we briefly mention related topics that we had no space to extensively analyze in the main text.

T – duality and target space supersymmetry: A similar clash between T-duality and supersymmetry on the target space takes place as well [3]. The conventional definition of a background with unbroken target space supersymmetry requires that solutions to the Killing spinor equations exist. In the presence of rotational-type Killing vectors abelian T-duality breaks manifest target space supersymmetry in the sense that no Killing spinors exist in the dual background [3,4]. The crucial difference with the extended world-sheet supersymmetry case is that here the effective field theory is not enough at all to understand

³ Notice that, there is a distinction between $+$ and the $-$ components below even though the torsion vanishes. This is a novel characteristic of non-local realizations of supersymmetry [2].

the fate of target space supersymmetry under duality, which nevertheless does not destroy the supersymmetry in a string setting. Finding non-local Killing spinors [5] is not adequate to generate the whole supersymmetry transformation. An approach to the problem using CFT concepts has been made in [6], where again the use of parafermions becomes necessary. Finally, let us mention that breaking of manifest target space supersymmetry occurs hand and hand with breaking of local $N = 4$ extended world-sheet supersymmetry. However, although in the latter case the $N = 2$ part remains local, the local target space supersymmetry completely breaks. This is attributed to the relation of Killing spinors and complex structures [23], i.e. $F_{\mu\nu} = \bar{\eta}\Gamma_{\mu\nu}\eta$, which makes possible to construct a local complex structure out of a non-local Killing spinor.

Theorems revised: We have seen that, in a string theoretical setting, non-local complex structures are equally acceptable as local ones, since they capture stringy effects that are manifest at the (super)CFT level. Hence, it is important to investigate the conditions under which there is non-locally realized extended supersymmetry in general, without preassuming its origin. This was done in [2]. In addition, all theorems that were proved in the past in the context of 2-dim supersymmetric σ -models with extended supersymmetry (see for instance [14] and references therein), always assuming local realizations, need to be revised, since they clearly fail in the presence of non-local realizations. For instance, $N = 4$ and zero torsion does not imply Ricci flatness (the reader can easily check that by computing the Ricci tensor for the simple metric in (2.23)). Steps towards this direction have been made [2].

Non – abelian T – duality and supersymmetry: Let us just consider the effect of the non-Abelian duality transformations on pure gravitational backgrounds with extended $N = 4$ world-sheet supersymmetry. For $SO(3)$ -invariant metrics, the complex structures either can be $SO(3)$ singlets, thus remaining invariant under the non-Abelian group action, $\mathcal{L}_J(F_I)_{\mu\nu} = 0$, or form an $SO(3)$ triplet when $\mathcal{L}_J(F_I)_{\mu\nu} = \epsilon_{JIK}(F_K)_{\mu\nu}$. The Eguchi–Hanson metric corresponds to the first case, while the Taub–NUT and the Atiyah–Hitchin metrics to the second [18]. It should be clear then that the dual version of the Eguchi–Hanson instanton with respect to $SO(3)$ will have an $N = 4$ world-sheet supersymmetry locally realized. On the other hand, performing the non-Abelian $SO(3)$ -duality to the Taub–NUT and the Atiyah–Hitchin metrics will result in a total loss of all the locally realized extended world-sheet supersymmetries [1]. We expect to have a non-local realization of supersymmetry in such cases with three non-local complex structures that satisfy the general conditions of [2]. There is now some work on non-Abelian duality and canonical transformations in the target space [24] which should be helpful in uncovering the hidden non-local supersymmetry these models are expected to have.

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